On the Numerical Solution of Linear Integral Equations.

By H. Bateman.

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§ 1. *Introduction.*

The numerical solution of linear integral equations of the types studied by Volterra has formed the subject of a recent memoir by E. T. Whittaker.[*] Numerical methods are needed also for the solution of the linear integral equation studied by Fredholm[†] and Hilbert[‡] and the object of this paper is to describe a method which may sometimes be useful.

The linear integral equation of the second kind,

$$f(s) = \phi(s) - \lambda \int_0^1 \kappa(s, t)\,\phi(t)\,dt, \tag{1}$$

may be solved for the unknown function $\phi(t)$ either directly or indirectly. In the direct methods of solution the function $\phi(t)$ is expressed by means of infinite series of terms involving repeated integrals. In the so-called method of Neumann only one infinite series is used, while in the more complete method of Fredholm the function $\phi(t)$ is expressed as the ratio of two infinite series which converge for all values of the parameter λ. The method of Neumann has been employed on many occasions to obtain numerical results and is generally more convenient to use than Fredholm's method on account of the great complexity of the expressions occurring in Fredholm's series. There are occasions, however, when the Neumann series fails to converge, or converges only slowly, and then some other method such as Fredholm's must be used. It seems desirable if possible to devise simple approximate methods which possess some of the advantages of Fredholm's method, because in many cases the evaluation of repeated integrals becomes very tedious and the use of the direct methods of solution becomes impracticable except for a rough approximation.

Of the indirect methods of solution, we shall be concerned here with one which depends on the use of an approximate expression for the kernel of the integral equation. E. Schmidt[§] has already sketched a method in which the kernel $\kappa(s, t)$ is replaced by an approximate expression

$$\bar{\kappa}(s, t) = \sum_{m=1}^{n} \alpha_m(s)\,\beta_m(t),$$

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[*] E. T. Whittaker, 'Roy. Soc. Proc.,' A, vol. 94, p. 367 (1918).

[†] I. Fredholm, 'Acta Math.,' vol. 27 (1903).

[‡] D. Hilbert, 'Gött. Nachr.,' 1904.

[§] E. Schmidt, 'Math. Ann.,' vol. 63, p. 433 (1907); vol. 64, p. 161 (1907).

consisting of a finite number, n, of products of functions of s and t. It is known that the solving function for this approximate kernel can be written down in finite terms* and the solution of the integral equation with this approximate kernel can be regarded as an approximate solution of the original integral equation. The success of the method evidently depends upon an appropriate choice of the functions $\alpha_m(s)$, $\beta_m(t)$. Schmidt has something to say on this matter; in particular, he indicates how the functions $\alpha_m(s)$, $\beta_m(t)$ must be chosen so as to make the double integral

$$\int_0^1\int_0^1 [\kappa(s, t) - \bar{\kappa}(s, t)]^2 \, ds \, dt$$

um. The required functions $\alpha_m(s)$, $\beta_m(t)$ turn out to be the first n normal orthogonal functions associated with the kernel $\kappa(s, t)$ in a manner described by Schmidt. These functions are generally difficult to find, consequently this best method of approximating to the kernel is generally useless for an easy numerical solution of the integral equation.

In the following paper we shall show that by making a different choice of the functions $\alpha_m(s)$, $\beta_m(t)$ we can obtain formulæ which are quite suitable for numerical work.

§ 2. *Method of Approximation.*

We shall use the term solving function to denote a function $K(s, t)$ such that the solution of the integral equation (1) may be expressed in the form

$$\phi(s) = f(s) + \lambda \int_0^1 K(s, t) f(t) \, dt \tag{2}$$

when $f(s)$ is any continuous function. Our method is now based on the following general theorem.†

Let $\kappa(s, t)$ be a kernel for which the solving function $K(s, t)$ is known, then if $f_1(s), f_2(s) \ldots f_n(s)$; $g_1(t), g_2(t) \ldots g_n(t)$ are given continuous functions and a_{lm} $(l, m = 1, 2, \ldots n)$ given constants, the solving function $H(s, t)$ for a kernel $h(s, t)$, defined by the equation

$$\begin{vmatrix} \kappa(s, t) - h(s, t) & f_1(s) & f_2(s) & \ldots\ldots & f_n(s) \\ g_1(t) & a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ g_2(t) & a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ g_n(t) & a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} = 0, \tag{3}$$

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* E. Goursat, 'Bull. Soc. Math. France,' vol. 35, p. 163 (1907); H. Lebesgue, *ibid.*, vol. 36, p. 3 (1908); E. Schmidt, *loc. cit.*

† 'Mess. of Math.,' April (1908).

is given by the equation

$$\begin{vmatrix} K(s,t)-H(s,t) & \phi_1(s) & \phi_2(s) & \ldots\ldots & \phi_n(s) \\ \chi_1(t) & A_{11} & A_{12} & \ldots\ldots & A_{1n} \\ \chi_2(t) & A_{21} & A_{22} & \ldots\ldots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \chi_n(t) & A_{n1} & A_{n2} & \ldots\ldots & A_{nn} \end{vmatrix} = 0, \tag{4}$$

where

$$\left.\begin{aligned} \phi_l(s) &= f_l(s)+\lambda\int_0^1 K(s,t)f_l(t)\,dt, \\ \chi_m(t) &= g_m(t)+\lambda\int_0^1 g_m(s)\,K(s,t)\,ds, \\ A_{lm} &= a_{lm}+\lambda\int_0^1 g_l(x)\,\phi_m(x)\,dx \end{aligned}\right\}. \tag{5}$$

The functions occurring in this theorem are all supposed to be completely defined in the intervals $(0 \leqq s \leqq 1)$, $(0 \leqq t \leqq 1)$ and to be continuous throughout these intervals.

We are interested chiefly in the case when $\kappa(s,t) = 0$, $K(s,t) = 0$, then $\phi_l(s) = f_l(s)$, $\chi_m(t) = g_m(t)$, and the value of A_{lm} is at once known.

Let us suppose that $w(s, t)$ is the kernel of an integral equation

$$\alpha(s) = \beta(s)-\lambda\int_0^1 w(s,t)\,\beta(t)\,dt \tag{6}$$

whose approximate solution is desired.

Putting

$$f_m(s) = -w(s, t_m), \qquad g_l(t) = -w(s_l, t), \qquad a_{lm} = -w(s_l, t_m), \tag{7}$$

where $s_1, s_2, \ldots s_n$; t_1, t_2, t_n are particular values of s and t belonging to the interval $(0, 1)$ it is clear that equation (3) becomes

$$\begin{vmatrix} h(s,t) & w(s,t_1) & w(s,t_2) & \ldots\ldots & w(s,t_n) \\ w(s_1,t) & w(s_1,t_1) & w(s_1,t_2) & \ldots\ldots & w(s_1,t_n) \\ w(s_2,t) & w(s_2,t_1) & w(s_2,t_2) & \ldots\ldots & w(s_2,t_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ w(s_n,t) & w(s_n,t_1) & w(s_n,t_2) & \ldots\ldots & w(s_n,t_n) \end{vmatrix} = 0. \tag{8}$$

This equation defines a function $h(s, t)$, which satisfies the conditions

$$\left.\begin{aligned} h(s,t_m) &= w(s,t_m) \qquad (0 \leqq s \leqq 1) \qquad (m = 1, 2 \ldots n) \\ h(s_l,t) &= w(s_l,t) \qquad (0 \leqq t \leqq 1) \qquad (l = 1, 2 \ldots n) \end{aligned}\right\}. \tag{9}$$

The function $h(s, t)$ thus agrees in value with the function $w(s, t)$ over a network of lines in the (s, t) plane; it thus represents a good approximation to the function $w(s, t)$. It follows, moreover, from formula (4) that the solving function $H(s, t)$ can be written down, and so an approximate solution of the integral equation (6) is furnished by the formula

$$\beta(s) = \alpha(s) + \lambda \int_0^1 H(s, t)\, \alpha(t)\, dt. \tag{10}$$

It should be noticed that the above approximate representation of $w(s, t)$ becomes exact when $w(s, t)$ is of the form

$$w(s, t) = \sum_{m=1}^{n} p_m(s)\, q_m(t),$$

it fails when $w(s, t)$ is the sum of less than n products of type $p(s)\,q(t)$, for then the co-factor of $h(s, t)$ in the determinant (3) vanishes. Approximate values of the singular values of λ for the integral equation may be obtained by equating to zero the coefficient of $H(s, t)$ in equation (4). It should be remarked that, when n is even, the approximate values of λ thus obtained may be complex even when the exact values of λ are all real.

§ 3. *Illustrative Example.*

Let us apply our method to Hilbert's integral equation, in which the kernel $w(s, t)$ is defined by the equations

$$w(s, t) = \begin{matrix} s(1-t) & s \leqq t \\ t(1-s) & s \geqq t \end{matrix}.$$

We have

$$A_{lm} = -w(s_l, t_m) + \lambda \int_0^1 w(s_l, x)\, w(x, t_m)\, dx.$$

Now
$$\int_0^1 w(s, x)\, w(x, t)\, dx = \begin{matrix} \tfrac{1}{6} s(1-t)(2t-s^2-t^2) & s \leqq t, \\ \tfrac{1}{6} t(1-s)(2s-s^2-t^2) & s \geqq t, \end{matrix}$$

hence the value of A_{lm} is easily found.

We shall interest ourselves simply in the way in which our method enables us to approximate to the singular values of λ, for which the homogeneous equation

$$\phi(s) = \lambda \int_0^1 w(s, t)\, \phi(t)\, dt$$

can be satisfied. It is known that the exact singular values of λ are π^2, $(2\pi)^2$, $(3\pi^2)$, ..., while, according to our method of approximation, they should be given approximately by the determinant $|A_{lm}| = 0$, the degree of approximation being better the larger the value of n. Let us take $n = 5$,

$$s_1 = t_1 = \tfrac{1}{6}, \quad s_2 = t_2 = \tfrac{2}{6}, \quad s_3 = t_3 = \tfrac{3}{6}, \quad s_4 = t_4 = \tfrac{4}{6}, \quad s_5 = t_5 = \tfrac{5}{6},$$

then if $\lambda = 216x$ the equation for λ may be written in the form

$$\begin{vmatrix} 5(1-10x) & 4(1-19x) & 3(1-26x) & 2(1-31x) & 1-34x \\ 4(1-19x) & 8(1-16x) & 6(1-23x) & 4(1-28x) & 2(1-31x) \\ 3(1-26x) & 6(1-23x) & 9(1-18x) & 6(1-23x) & 3(1-26x) \\ 2(1-31x) & 4(1-28x) & 6(1-23x) & 8(1-16x) & 4(1-19x) \\ 1-34x & 2(1-31x) & 3(1-26x) & 4(1-19x) & 5(1-10x) \end{vmatrix} = 0.$$

The determinant is easily evaluated by repeatedly subtracting rows and columns and we obtain the equation

$$130x^5-441x^4+488x^3-206x^2+30x-1 = 0,$$

i.e., $$(x-1)(2x-1)(5x-1)(13x^2-22x+1) = 0.$$

Approximate singular values of λ are thus

$$\lambda_1 = 10{\cdot}09, \quad \lambda_2 = 43{\cdot}2, \quad \lambda_3 = 108, \quad \lambda_4 = 216, \quad \lambda_5 = 355{\cdot}2.$$

The first is equal to π^2 within about 2 per cent., the second differs from $4\pi^2$ by about 9 per cent. while the errors in the other cases are larger still. If with $n = 3$ we take

$$s_1 = t_1 = \tfrac{1}{4}, \quad s_2 = t_2 = \tfrac{1}{2}, \quad s_3 = t_3 = \tfrac{3}{4},$$

we shall find the approximate values

$$\lambda_1 = 10{\cdot}37, \quad \lambda_2 = 48{\cdot}13, \quad \lambda_3 = 126{\cdot}13.$$

With $n = 1$ and $s_1 = t_1 = \frac{1}{2}$ we find simply the approximate value

$$\lambda_1 = 12.$$

In the case when $n = 5$, the percentage error in the value of λ is roughly proportional to λ. When we are dealing with the solution of the equation (6) instead of the homogeneous equation, we can generally expect the percentage error to increase with λ. If we wish our answer to be correct within 1 per cent. when the value of λ is in the neighbourhood of the first singular value, we should take $n = 7$ at least, with a value of λ equal to about half the first singular value, it may be sufficient to take $n = 5$, for values of λ larger than the first singular value, still higher values of n, such as 9 or 11, may be needed to give the desired degree of approximation.

§ 4. *Polynomial Kernel.*

The method of § 2 can be extended without difficulty to the case of a system of integral equations by making use of Fredholm's well known device. It may also be extended in other directions; we may, for instance, write

$$f_m(s) = \kappa(s, t_m) - w(s, t_m), \qquad g_l(t) = \kappa(s_l, t) - w(s_l, t),$$

$$a_{lm} = \kappa(s_l, t_m) - w(s_l, t_m),$$

where $\kappa(s, t)$ is a kernel for which the solving function $K(s, t)$ is known. The function $h(s, t)$ is then equal to $w(s, t) - k(s, t)$ over the lines of a network.

Another method depends on the use of an approximate polynomial representation of the kernel $w(s, t)$. Let us now put

$$\kappa(s, t) = 0, \qquad f_m(s) = s^m, \qquad g_m(t) = t^m, \qquad K(s, t) = 0,$$

then
$$A_{lm} = a_{lm} + \lambda \int_0^1 x^{l+m}\,dx = a_{lm} + \frac{\lambda}{l+m+1}.$$

If now the expression for $h(s, t)$ is

$$h(s, t) = -\sum_{r=0}^{n} \sum_{m=0}^{n} b_{mr} s^r t^m,$$

where
$$b_{mr} = 0, \qquad m + r > n,$$

we may determine the constants a_{lm} by solving the equations

$$\begin{aligned} x_1 &= b_{11}y_1 + b_{21}y_2 + \ldots b_{n1}y_n, \\ x_2 &= b_{12}y_1 + b_{22}y_2 + \ldots b_{n-1,2}\, y_{n-1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= b_{1n}y_1, \end{aligned}$$

the solutions being

$$\begin{aligned} \Delta y_1 &= a_{1n}x_n + a_{1,\,n-1}x_{n-1} + \ldots a_{11}x_1, \\ \Delta y_2 &= a_{2n}x_n + a_{2,\,n-2}x_{n-1} + \ldots a_{21}x_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

$$\Delta = \begin{vmatrix} b_{11} & b_{21} \;\ldots\ldots\; b_{n1} \\ b_{12} & \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots \end{vmatrix}$$

It is clear from these equations that $a_{mr} = 0$, $m + r < n + 1$, also

$$a_{1n} = \frac{\Delta}{b_{1n}}, \qquad a_{2,\,n-1} = \frac{\Delta}{b_{2,\,n-1}}, \qquad a_{2,\,n} = -\frac{\Delta b_{1,\,n-1}}{b_{1n} b_{2,\,n-1}}.$$

Our object now is to obtain an approximate representation of a kernel $\kappa(s, t)$ by means of a polynomial in s and t, or by some other double series of functions of s and t.

Some interpolation formulæ for functions of two variables have been studied by S. Namuri.* If we write

$$\Delta_x^l f(x+h, y) - \Delta_x^l f(x, y) = \Delta_x^{l+1} f(x, y),$$
$$\Delta_y^m \Delta_x^l f(x, y+k) - \Delta_y^m \Delta_x^l f(x, y) = \Delta_y^{m+1} \Delta_x^l f(x, y),$$

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* S. Namuri, 'The Tôhoku Mathematical Journal,' vol. 18, p. 309 (1920).

for all values of 1, m 0, i, 2, ... , it is easy to see that the operators Δ_x and Δ_y are permutable, in other words we have the relation

$$\Delta_y^m \Delta_x^l f(x, y) = \Delta_x^l \Delta_y^m f(x, y)$$

for all values of y. Introducing the notation

$$\binom{s}{k} = \frac{s(s-1) \dots (s-k+1)}{1 \,.\, 2 \dots k},$$

the desired interpolation formula may be written

$$f(a+sh,\, b+tk) = \sum_{l=0}^{\mu-1} \sum_{m=0}^{\nu-1} \binom{s}{l} \binom{t}{m} \Delta_x^l \Delta_y^m f(a, b) + \mathrm{R}_{\mu\nu},$$

$$x = a+sh, \qquad y = b+tk,$$

where the remainder $\mathrm{R}_{\mu\nu}$ has the form

$$\binom{s}{\mu} h^\mu \frac{\partial^\mu}{\partial \alpha^\mu} f(\alpha, y) + \binom{t}{\nu} k^\nu \frac{\partial^\nu}{\partial \beta^\nu} f(x, \beta) - \binom{s}{\mu} \binom{t}{\nu} h^\mu k^\nu \frac{\partial^{\mu+\nu}}{\partial \alpha^\mu \partial \beta^\nu} f(\alpha, \beta),$$

where α lies between the least and greatest of the quantities

$$a,\ a+h,\ a+2h\ \dots, \qquad a+(\mu-1)h,\ x,$$

and β lies between the least and greatest of the quantities

$$b,\ b+k,\ b+2k\ \dots, \qquad b+(\nu-1)k,\ y.$$

If the function $f(x, y)$ is actually a polynomial of degree $\mu-1$ in x and of degree $\nu-1$ in y, the remainder vanishes.

The above method of interpolation is suitable for the case in which the function $f(x, y)$ is known at the points of intersection of two sets of lines parallel to the axes of co-ordinates. Mamuri has extended the formula for the case in which the interval between two successive lines is not constant, but it is desirable to have a formula which is applicable when the points at which $f(x, y)$ is known are distributed at random.

Let us consider the simple case in which there are six points, P_1, P_2, P_3, P_4, P_5, P_6, and let $\mathrm{C}_r = 0$ be the equation of the conic through all the points except P_r, also let c_r be the value of C_r when the co-ordinates of P_r are substituted in C_r, then if p_r denotes the value of $f(x, y)$ at the point P_r, the required formula is

$$f(x, y) \sim \sum_{r=1}^{6} p_r \frac{\mathrm{C}_r}{c_r}.$$

When the six points are the six intersections of four lines $\mathrm{L}_1 = 0$, $\mathrm{L}_2 = 0$, $\mathrm{L}_3 = 0$, $\mathrm{L}_4 = 0$, we may write

$$\mathrm{C}_1 = \mathrm{L}_2\mathrm{L}_3, \quad \mathrm{C}_2 = \mathrm{L}_3\mathrm{L}_1, \quad \mathrm{C}_3 = \mathrm{L}_1\mathrm{L}_2, \quad \mathrm{C}_4 = \mathrm{L}_1\mathrm{L}_4, \quad \mathrm{C}_5 = \mathrm{L}_2\mathrm{L}_4, \quad \mathrm{C}_6 = \mathrm{L}_3\mathrm{L}_4.$$

These two methods, which are well known, may be extended, for when there are $\frac{1}{2}n(n+1)$ points at which the value of $f(x,y)$ is known, we may use a curve of the $(n-1)$th degree instead of a conic, and when the $\frac{1}{2}n(n+1)$ points lie at the intersection of $n+1$ lines we may represent each of these curves as the product of $(n-1)$ lines.

§ 5. *Possible Applications.*

It is known that many problems in the theory of elasticity may be reduced to an integral equation of the second kind or to a system of integral equations of the second kind.[*] In particular, Fredholm has indicated a method of reducing a two-dimensional elastic problem to a pair of integral equations with kernels that do not become infinite within the range of integration. The torsional problem in the theory of elasticity[†] and many electrical problems are equivalent to well known potential problems that are reducible to integral equations of the second kind. Certain problems relating to the flexure of beams are likewise reducible to integral equations.

It is thought that the method of § 2 may be sometimes useful in dealing with such problems. It is true that the work will be laborious, but this seems unavoidable when a numerical solution is desired.

Problems relating to the critical strength of struts and structures, the vibrations of rods and membranes, the whirling of shafts, etc., may also be treated by this method, the differential equations being replaced by integral equations. It is thought, however, that in these cases the methods of approximation proposed recently by other writers[‡] may be more convenient to use.

In the case of the equation of a vibrating membrane

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+k^2 V = 0,$$

we may employ Whittaker's solution[§]

$$V = \int_0^{2\pi} e^{ik(x\cos\alpha+y\sin\alpha)}\,\phi(\alpha)\,d\alpha.$$

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[*] See, for instance, E. and F. Cosserat, 'Comptes Rendus,' Paris, vol. 133 (1901); I. Fredholm, 'Arkiv för Math. och Fys.,' vol. 2 (1906); G. Lauricella, 'Atti Lincei' (5), vol. 15_1, pp. 426, 610, vol. 15_2, pp. 75–83 (1906); 'Il Nuovo Cimento' (5), vol. 13 (1907), four papers; 'Rend. Lincei' (1907); 'Acta Math.' (1908–9); T. Boggio, 'Atti Lincei,' vol. 16_2, pp. 248, 441 (1907); Marcolongo, 'Toulouse Ann.' (1908); J. Hadamard, 'Mémoires de l'Institut' (1908); A. Korn, 'Ann. École Normale' (3), vol. 25 (1908); Zaremba, *ibid.*, p. 337; A. Haar, 'Gött. Nach.' (1907); O. Tedone, 'Rend. Lincei,' p. 232 (1904); C. E. Weatherburn, 'Phil. Mag.,' vol. 32, p. 15 (1916); 'Mess. of Math.,' vol. 47, p. 114 (1917).

[†] *Cf.* L. Bairstow, 'Roy. Soc. Proc.,' A, vol. 95, p. 457 (1919).

[‡] W. L. Cowley and H. Levy, 'Roy. Soc. Proc.,' A, vol. 95, p. 440 (1919); 'Phil. Mag.,' vol. 41, p. 584 (1921); A. A. Griffith, 'National Advisory Committee for Aeronautics,' R. and M. 545 (1918–19); R. V. Southwell, 'Phil. Mag.,' vol. 41, p. 419 (1921).

[§] 'Math. Ann.' (1903); Whittaker-Watson, "Modern Analysis."

If the boundary condition is

$$V = 0 \text{ when } x = r(\theta)\cos\theta, \qquad y = r(\theta)\sin\theta,$$

we obtain the integral equation

$$0 = \int_0^{2\pi} e^{ikr(\theta)\cos(\theta-\alpha)}\,\phi(\alpha)\,d\alpha,$$

which may be regarded as a limiting case of the homogeneous equation

$$\mu(k)\,\phi(\theta) = \int_0^{2\pi} e^{ikr(\theta)\cos(\theta-\alpha)}\,\phi(\alpha)\,d\alpha.$$

We may apply our method of approximation to obtain the possible values of the functions $\mu(k)$. Equating these to zero we obtain the possible values of k.

The Atomic Process in Ferromagnetic Induction.

By Sir J. Alfred Ewing, K.C.B., F.R.S.

(Received November 29, 1921.)

The object of this paper is to amend, in an important particular, the theory of ferromagnetic induction put forward by me more than 30 years ago,* and to describe a new model. That theory was itself a modification of the earlier theory of Weber. To Weber is due the fundamental notion that a substance contains minute particles, each of which acts as a magnet, and that in the process of magnetising a ferromagnetic substance these are turned into more or less complete alignment. The ultimate magnetic particles used to be called "molecular magnets": we now recognise them as attributes of the atom, not of the molecule, and (in all probability) they derive their magnetic moment from the circulation of electricity in electron orbits or in ring electrons. What turns is not the molecule nor the atom, but something within the atom.

The characteristics which distinguish ferromagnetic substances from other paramagnetics are: (1) the much larger amount of magnetism they can acquire under the action of an impressed field; (2) the fact that the acquired magnetism tends towards a saturation limit when the field is progressively increased; (3) the fact that the acquired magnetism shows hysteresis with

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* "Contributions to the Molecular Theory of Induced Magnetism," 'Roy. Soc. Proc.' vol. 48, p. 342 (1890); 'Phil. Mag.,' September, 1890. See also 'Magnetic Induction in Iron and other Metals,' Chap. XI.